Using Jack's Work to Introduce Notation for Multiplication of Fractions

This story is a part of the series:

***What's Next?* Stories of Teachers Engaging in Collaborative Inquiry Focused on Using Student Thinking to Inform Instructional Decisions**

What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

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Introduction

A teacher was working on helping her students to understand division with whole numbers and fractions. She assessed all the students in her class by posing a measurement division word problem with a whole number as the dividend and a unit fraction as the divisor. After analyzing the students' work, she identified near-term learning goals for her students and developed a new mathematics problem of the same type. This second time, she posed the problem to the whole class and deliberately selected students to share their work in an effort help students learn to communicate their solution strategies and ideas in writing.

Relevant Florida Mathematics Standards

MAFS.5.NF.2.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) x 4 = 1/3.

b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 x (1/5) = 4.

c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?

Background Information

The interested reader may want to read chapter one of *Extending Children's Mathematics: Fractions and Decimals* (Empson & Levi, 2011). This chapter provides a more detailed explanation of equal-sharing problems and the strategies that children typically use when solving them.

Empson, S. & Levi, L. (2011). *Extending Children's Mathematics: Fractions and Decimals.* Portsmouth, NH: Heinemann.

Analyzing Student Thinking

The teacher began this lesson by providing all of her students with the following measurement division problem. The students were asked to solve the problem in any way that made sense to them. They had access to writing materials, linking cubes, and several other manipulatives. The teacher encouraged the students to record their thinking in a way such that someone else could understand their thinking without any further explanation.

> *Ms. Jones has six sandwiches. She wants to give each of her students 1/4 of a sandwich for snack. How many students can Ms. Jones give a snack to?*

After the students completed the problem, the teacher collected their written work and sorted the work according to strategies students used to solve the problem.[1]

Strategies Used by Students in the Class

The student who uses a *direct modeling* strategy generates a solution to a story problem or number fact by representing each number in the problem using manipulatives (including fingers)

1 The descriptions of strategies presented in this section are the current descriptions used by our team, and we consider them to be fluid, because our understanding of these ideas continues to evolve. For a more detailed discussion of these terms, consider reading Empson, S. & Levi, L. (2011).

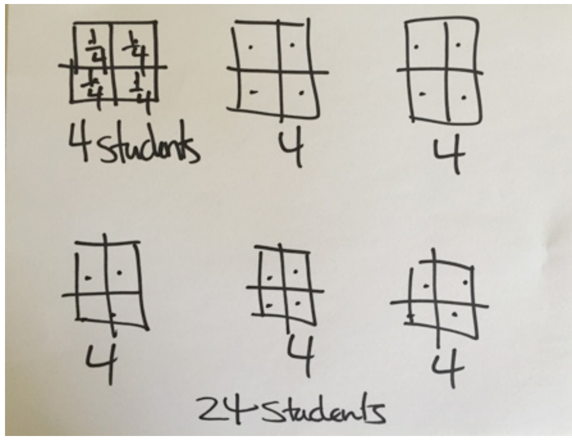

Figure 1. A ***direct modeling*** *solution to the sandwich problem.*

or by drawing a model on paper. For this problem, the student would probably draw representations of six sandwiches. The student would then divide each sandwich into fourths and count the 1/4-sized pieces in the six sandwiches. See Figure 1 for an example of a student using this strategy.

The student who uses a *counting* strategy uses a problem-solving approach that does not require representing all quantities in the problem concretely. The student has learned that a number can be stated rather than represented concretely. In this problem, the student might count by fours, using each count of four to represent one sandwich, six times to get 24.

The student who uses a *known fact* strategy recalls the answer from memory. For this problem, the student would say that he or she knows that $6 \times 4 = 24$, so the teacher could serve 24 students one fourth of a sandwich each.

The students who attempted to use a strategy that does not clearly fit in to one of the categories listed above were placed in the *other* category.

Figure 2 shows the results of the classification.

The teacher noticed the large number of students who fell into the *direct modeling* category. Most of her students used a visual model to represent the structure of the story problem. The teacher decided to provide the students with a similar measurement division word problem and lead a discussion around how to use conventional mathematics notation to communicate their thinking. She thought this discussion might help prompt students to use more advanced notation and strategies with understanding.

The teacher developed the following near-term goal for her students.

> *When solving measurement division problems involving fractional pieces, students will learn to express their* ***direct modeling*** *strate-*

Direct modeling	*Counting*	*Known fact*	*Other*
Jillian	Divya	Ava	Wyatt
Bailey	Noah	Justin	Cecilia
Audrey	Conor		
Shannon*			
Sheldon			
Jack			
Roman			
Kendall			
Andrew			

**Incorrect solution*

Figure 2. The sorting of the students by strategy used on the sandwich problem.

gies with a more sophisticated and abstract notation that follows conventional rules of notation in mathematics.

Planning for the Lesson

The teacher considered a variety of problems and decided to use the following measurement division word problem for her next problem to give to students.

Mr. Vernon has five bars of clay. He wants to make bowls from the clay. It takes 1/3 of a bar of clay to make one bowl. How many bowls can he make?

Rationale for the problem selected

The teacher thought that another measurement division problem would encourage many students to use a *direct modeling* strategy again. She saw this result as advantageous, because it would provide opportunities for her to present formal mathematical notation that can be used to express the ideas the students were expressing in pictorial representations.

Notes on what to notice about student thinking

The teacher considered how her students previously solved the problem and anticipated the following four strategies.

1. The student draws a picture of five bars of clay, splits each into thirds, and then counts each small piece by ones from 1 to 15. (*direct modeling*)

2. The student draws a picture of five bars of clay, splits each into thirds, and then labels each third with *1/3*. Then counts by threes to get to 15. (*direct modeling*, but moving toward a *counting* strategy)

3. The student writes or says five bars of clay and then notes that each bowl takes *1/3* of a bar. The student writes or says, "Every three pieces makes one bowl, so 3, 6, 9, 12, 15 (lifting one finger for each count)" and determines that he or she can make 15 bowls. (*counting*)

4. The student knows to multiply 3 × 5 to arrive at the answer of 15 bowls. (*known fact*)

The teacher was hoping to encourage her students to begin seeing that three groups of 1/3 is one, six groups of 1/3 is two, nine groups of 1/3 is three, etc. She also wanted them to see this notated as 3 x 1/3 = 1, 6 x 1/3 = 2, 9 x 1/3 = 3, etc. The teacher considered these four general strategies and decided to look for students who used the first two strategies, which clearly connect thirds to this notation she wanted her students to begin using.

The teacher thought about how she might be able to fully engage all students in this task to set them up for success. She decided that she would initially put focus her effort on Wyatt, Cecilia, and Shannon, because they fell into the "*other*" category on the first problem or arrived at an incorrect solution. She chose to spend the beginning of the individual work time talking with these three students to ensure that they understood the problem itself and had decided on a good strategy for trying to solve the problem.

After making sure those three students were off to a good start, the teacher decided she would turn her attention toward working with the students who used *direct modeling* strategy in the previous problem to try to find students who used *direct modeling* strategies (numbered 1 and 2 in the above discussion) on the new problem. She wanted these two students to share their thinking with the class during the closing discussion so that she could show them how to use conventional mathematical notation to express their ideas.

Lesson Plan

During the Analyzing Student Thinking phase, the teacher developed the following goal.

When solving measurement division problems involving fractional pieces, students will learn to express their ***direct modeling*** *strategies with a more sophisticated and abstract* *notation that follows conventional rules of notation in mathematics.*

The following lesson plan was implemented after the initial information was collected on the students and was based upon the goal stated above.

1. Provide the students with the following problem:

   a. *Mr. Vernon has five bars of clay. He wants to make bowls from the clay. It takes 1/3 of a bar of clay to make one bowl. How many bowls can he make?*

   b. Display the problem for the students to see, or type it at the top of a page with plenty of space in which students can record their thinking.

2. Read the problem aloud to the students and confirm that they understand the context and what they are being asked to do.

3. Ask the students to solve the problem in whatever way makes the most sense to them. Provide access to writing materials and manipulatives, but emphasize that they are not required to use them. Remind the students to record their thinking so that someone else could make sense of their work without any further explanation. Encourage the students who quickly solve the problem in one way to look for another way to solve it.

   a. As an alternative, consider posing a version of the problem without numbers. Doing this can help students focus on the salient aspects of the problem without being encumbered by thinking about the numbers. Introduce the numbers after discussing the context.

   b. If you choose this option, begin by reading the problem without the numbers and inserting the word "some" and "some part" for the whole number and fraction respectively. Then ask the students, "What things do you need to know to find out how many bowls of clay Mr. Vernon can make?"

   c. As the discussion unfolds, gradually release pieces of information in this order: *He has five bars of clay. Each bowl requires 1/3 of a bar.*

   d. Retell the problem to the class, this time using the numbers five and 1/3.

   e. Invite the class to solve the problem in any way that makes sense to them. Remind the students to record their thinking so that someone else could make sense of their work without any further explanation, and encourage the students who quickly solve the problem in one way to look for another way to solve it.

4. Be sure that students have access to writing materials while they work. Some may also find linking cubes useful in explaining their work. Consider making these available (but not mandatory) for students to use. While the students work, circulate to watch for students who use *direct modeling* strategies as described in Phase 3.

5. Look for students who are using the *direct modeling* strategies numbered 1 and 2 in the previous section. After finding them, ask them if they would be willing to share their thinking during the closing portion of the lesson. This will provide those students with time to begin thinking about what they would like to say in reference to their work.

6. If you find students who are struggling to solve the problem correctly, consider trying one of the following strategies:

   a. Remind the student of one of the strategies previously shared by his or her peers, and suggest trying one of those ways.

   b. Ask the student a more general question such as, "Do you know another way to think about this problem" or "What things could you consider before you start drawing the bars of clay?”

7. When you find students who have correctly solved the problem during this phase, consider trying one of the following strategies:

   a. Ask such students to find another way to solve the problem.

   b. Ask such students to consider other ways to represent their answers.

8. Invite the two students you identified during the work time to share their thinking. As they do so, attach notation to their work. An example is shown in Figure 3.

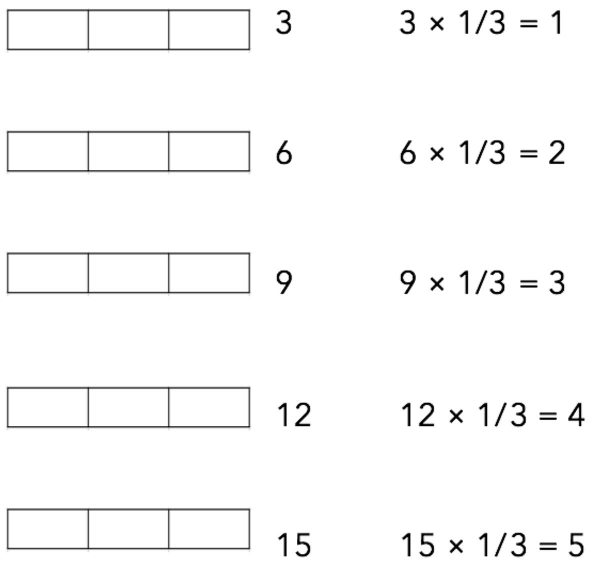

Figure 3. Recording notation to match a student’s strategy.

9. As you are recording this notation, be sure to encourage discussion around two questions:

   a. Where are the bars of clay, and where are the bowls in the picture and the notation?

   b. What pattern is emerging in the equations? The pattern begins with students noticing that, for every three bowls made, one bar of clay is used. From there, students can see that each time a multiple of three is multiplied by 1/3, a whole number of bars of clay is used.

10. If time permits, encourage students to explore the problem with different numbers to find out whether similar patterns emerge. For example, *Mr. Vernon has four bars of clay. He wants to make bowls from the clay. Each bowl takes 1/5 of a bar. How many bowls can he make?*

11. Record the notation for these new numbers in the manner on display in Figure 4. The students may notice that similar patterns emerge.

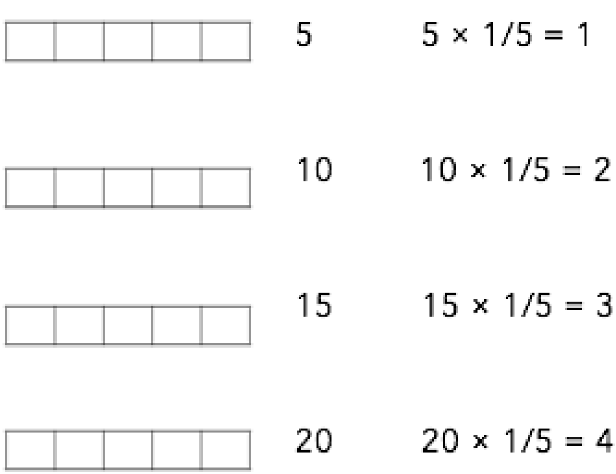

Figure 4. Recording notation to match a student’s strategy for the follow up problem.

Reflection

The teacher in the lesson identified two students to share their thinking. The first was Jack. Jack's work represented the *direct modeling* structure discussed in Planning for the Lesson section and is seen in Figure 5. He first drew a picture of five bars of clay, split each bar of clay into three equal-size pieces, and then counted 3, 6, 9, 12, 15.

The teacher asked Jack, "What does this 3, 6, 9, 12, 15 mean?" Jack replied by saying, "The three bowls you can make from this block of clay." The teacher then represented this as 3 × 1/3 = 1 as seen in Figure 5.

...offering symbolic notation based on students' drawings is helpful in making the students' thought processes more transparent and clear for other students as well as helping them to learn conventions in mathematical notation.

The teacher then asked the planned question, "What does this mean in our story?" The class agreed, it showed that three bowls could be made from each bar of clay.

The teacher then asked, "What might Jack write next?" The class agreed that Jack should write 6 × 1/3 = 2. The teacher again asked, "What does that mean in our situation?" The class agreed that this meant that six one-third sized bowls could be made from two blocks of clay.

The teacher continued the pattern with 3 × 1/3 = 9, 4 × 1/3 = 12, and 5 × 1/3 = 15; the result can be seen in Figure 5.

The second student who shared, Divya, used a more sophisticated strategy to represent her thinking.

To solve this problem, Divya wrote the following on her paper:

3 × 1/3 = 1
5 × 3 = 15

The teacher recorded Divya's work on chart paper and asked the class, "What do each of these numbers mean in the story, and how did Divya know to do that?" One student said, "Five is the five bars of clay, and the three is how much she split the bar into… …It's also the denominator of the fraction." A second student replied, "I agree that the five represents the five bars, but the three is from the three bowls that can be made from each bar." The teacher asked, "How are these explanations alike, and how are they different?" The class agreed that both explanations were ways to describe what each of the numbers Divya wrote represented.

After the discussion about Jack's strategy, Divya's strategy helped many students to see how they could think about measurement division problems involving unit fractions as multiplication and how that could be represented as a series of equations.

Lessons to Remember

The teacher considered the use of notation as it played out in this lesson. She concluded that offering symbolic notation based on students' drawings is helpful in making the students' thought processes more transparent and clear for other students as well as helping them to learn conventions in mathematical notation. She decided to continue using mathematical notation when students were sharing their *direct modeling* strategies as a way to help move students toward more sophisticated strategies and more abstract notation.

She also realized how much mathematics and how many mathematical ideas can be pulled from one problem. She planned to continue to use measurement division problems with whole numbers and unit fractions to help students make sense of division of a whole number by a fraction.

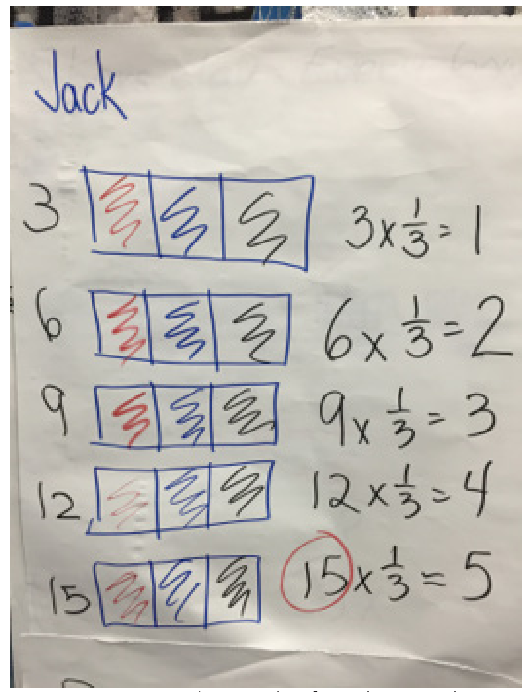

Figure 5. Jack's work after the teacher added the notation.

What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

What's Next? is a collection of stories documenting professional development experiences shared by elementary teachers working collaboratively to study the complex process of teaching and learning mathematics. Each story in the collection describes practicing teachers studying the thinking processes of real students and using what they learn about those students to make decisions and try to help advance those students' understanding on that day.

The teachers in each story start by learning about how individual students are solving a set of mathematics problems. They use this freshly gathered knowledge of student thinking to develop near-term learning goals for students and a lesson plan tailored to specific students on that specific day. One of the teachers implements the planned lesson while the other teachers observe in real time. The teachers then gather to discuss and reflect on their observations and insights.

In these lessons, the practice of teaching is slowed way down. The stories tell of teachers who are studying student thinking and using that information to plan and implement instructional decisions at a pace that is much slower than it occurs in daily practice. The stories in this collection also depict many aspects in common with formative assessment and lesson study, both of which are a process and not an outcome.

The stories depict real situations that occurred in real time and include both successes and shortcomings. We hope that the stories may be studied and discussed by interested educators so that the lessons and ideas experiences of these teachers and instructional coaches may contribute to additional learning and sharing among other interested teachers.

Learn more about these and other stories at **http://www.teachingisproblemsolving.org/**