Moving Students From Direct Modeling and Counting Strategies to Using Derived Facts When Solving Addition Problems

This story is a part of the series:

What's Next? **Stories of Teachers Engaging in Collaborative Inquiry Focused on Using Student Thinking to Inform Instructional Decisions**

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Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

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Introduction

During a professional development experience, a group of teachers conducted interviews with individual students in a second-grade class to examine their thinking as they were solving basic addition problems. Approximately half the class used *direct modeling* or counting strategies to evaluate the expressions. The teachers set the goal of having more students use more abstract, fact-based strategies to perform addition on single-digit numbers. They designed and implemented a lesson based around a progression of interrelated addition facts with the intent of moving students from *direct modeling* and counting strategies to use of derived facts.

Relevant Florida Mathematics Standards

MAFS.2.NBT.2.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

MAFS.1.OA.3.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Background Information

Consider reading chapter three in *Children's Mathematics: Cognitively Guided Instruction* (Carpenter et al., 2015). This chapter provides background on the varied ways children solve addition and subtraction problems, including the use of Derived Fact and Recall of Number strategies. It also expands upon the strategies explained in the Analyzing Student Thinking section of the present document.

An alternate source for information on fact fluency is the article "Fluency with basic addition" (Kling, 2011), which discusses components of fluency and promotes the development of fact strategies that leverage knowledge of relationships among facts.

For more information on assessing fact fluency, consider reading "Assessing basic fact fluency" (Kling & Bay-Williams, 2014), which provides a variety of ways to assess students' fluency with basic math facts.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's Mathematics: Cognitively Guided Instruction (2nd. Ed.)*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School*. Portsmouth, NH: Heinemann.

Kling, G. (2011). Fluency with basic addition. *Teaching Children Mathematics, 18*(2), 80–88.

Analyzing Student Thinking

In the context of a professional-development workshop near the beginning of the school year, a group of teachers conducted short, one-on-one interviews with each individual student in a second-grade class. The purpose of the interviews was to gain insight into the second graders' fluency with basic addition facts. The interviews involved verbally posing a series of addition problems one at a time. If the students' strategies were not readily apparent, the interviewer asked the students to explain how they arrived at the answer to each problem.

Fact Fluency Items

Students were asked to evaluate each one of the items in Figure 1, from the top to the bottom of the left-hand column and then from the top to the bottom of the right-hand column. As needed, the interviewer was allowed the flexibility to skip any problem(s) judged to be too challenging for the student. During the interview, the students did not have access to a variety of manipulatives (e.g., linking cubes, base-ten blocks) but paper and pencil and fingers were permitted.

As students responded to each item, the interviewer made note of the details of the strategy used (e.g., for 8 + 3, the student said "9, 10, 11" and raised one finger for each number word said aloud).

Named Strategies Commonly Used to Solve Single-digit Addition Problems

After the interview, the teachers reflected on the students' strategies using various named categories: *direct modeling*, *counting on from first*, counting on from larger, *fact recall—derived facts*, and *fact recall—known facts*[1]. These strategies are explained in more detail below.

A student using a *direct modeling* strategy to solve the problem represents each and every quantity in the problem with some sort of object (e.g., manipulatives, fingers, drawings). For example, when solving 6 + 8, the student who directly models will create a set of six objects and a set of eight objects and will then determine the answer by counting all the objects, starting at one, or will count forward, starting at the number in one of the sets.

A student using a *counting on from first* strategy determines the sum by counting forward from the first number presented in the problem. For the problem 6 + 8, this student might hold the quantity six mentally and count forward eight counts (from seven to 14), usually keeping track of the counts with fingers, tally marks, or objects.

A student using a counting on from larger strategy determines the sum by counting on from the larger number presented in the problem. For the problem 6 + 8, the student would hold the quantity eight mentally and count on six counts (from nine to 14), usually keeping track of the counts with fingers, tally marks, or objects. This strategy suggests the student has some understanding of the commutative property of addition (a + b = b + a), and uses that knowledge to solve the problem more efficiently.

5 + 5	7 + 7
7 + 3	7 + 5
8 + 3	7 + 6
2 + 7	9 + 7
6 + 4	5 + 7
5 + 6	6 + 8
4 + 8	8 + 9
2 + 8	6 + 9
6 + 6	8 + 7

Figure 1. Set of items used in the fact-fluency interview.

A student using *fact recall—derived facts*[2] uses a related, known fact to help solve a problem involving an unknown fact. For example, when solving 6 + 8, the student might decompose the eight into four and four, add the first four to six to get 10, and then add the remaining four to the 10 to get 14. In this case, the student is using the known facts 6 + 4 = 10, 8 = 4 + 4, and 10 + 4 = 14 to derive the unknown fact 6 + 8 = 14.

A *fact recall—known fact* strategy involves the student's recalling the relevant fact directly from memory. Answers are typically provided quickly. When asked how they got the answer, students often respond that they "just knew" the answer.

1 The descriptions of strategies presented here are the current descriptions used by our team, and we consider them open to change as our understanding of these ideas continues to evolve. For a more detailed discussion of these terms, consider reading Carpenter et al. (2015).

2 Students may use many other strategies that involve derived facts while solving number-fact problems. For more information about *derived facts*, read chapter three of Carpenter et al. (2015).

Examining the Strategies Used by Students in This Class

After the teachers analyzed students' strategies for each addition-fact problem in relation to the categories, they created a chart summarizing their findings (Figure 2). They recorded the most prevalent strategy used by every student, recognizing that most students used more than one strategy type during the interview.

About half of the students solved most of the problems using *direct modeling* or *counting* strategies. Therefore, the teachers wanted students to move toward using doubles facts as a benchmark and developed the following learning goal:

> *Students will derive addition facts by using doubles as benchmarks to evaluate addition expressions.*

The group of teachers decided that the second-grade students who used fingers to count or used *direct modeling* should be starting to use known facts to be more efficient. About half of the students demonstrated in interviews they could use ten as a benchmark or use known doubles facts to solve the problem. The teachers decided they would provide these students with opportunities to practice communicating their ideas, and the other students could learn from their peers.

Planning for the Lesson

The team worked collaboratively to develop a carefully planned sequence of addition facts to use in the lesson.

3 + 3
3 + 4
4 + 4
4 + 5
5 + 6
5 + 7
7 + 5
7 + 6
9 + 8

Rationale for the problems selected

The teachers wanted to pose a sequence of addition problems that would provide opportunities for students to use known facts to derive unknown facts. The first problem, 3 + 3, was chosen because most students knew that 3 + 3= 6 at a recall level. The next fact, 3 + 4, was selected to determine whether students would think relationally and recognize that the sum of the second expression would be one larger than the sum of 3 + 3. The expressions 4 + 4 and 4 + 5 were then selected to give a chance for students to start to generalize this idea. The expression 5 + 6 was chosen next to determine whether some stu-

Direct modeling	*Counting*	*Derived facts*		
		10 as a benchmark	*Doubles as a benchmark*	*Not sure of the relationship*
Jalen	Curtis	Braylon	Braylon	
Sabra Anne	Cody	Nicole	Nicole	
Dustin	Tyler (fingers)	Sophia	Sophia	
Quincy	Brandon	Joseph	Joseph	
	Abigail	Tyler	Codi	
	Isabella		Curtis	
	Nicole		Jayden	
	Jade			
	Katie			

Figure 2. Classification of the students' strategies for all the addition facts posed.

dents who were previously using *direct modeling* or *counting* strategies would think of relating 5 + 6 to 5 + 5 on their own.

The anticipated strategies relating doubles facts to doubles-plus-one facts are based on an intuitive understanding of the associative property of addition. The problem 7 + 5 was prepared to determine, if time allowed, whether students would use strategies based on the associative property of addition (e.g., 5 + (6 + 1) = (5 + 6) +1) or if they would use a strategy based on the commutative property of addition. Next, 7 + 6 would be given to determine whether students would relate this problem to the one before. The 9 + 8 expression was selected to be used as a challenging problem, but it would only be used if time allowed it and if the teacher thought the students would benefit from trying to solve that one with their new skills.

Strategy for differentiation to meet the needs of all students in the class

Deliberately selected students would be called upon to share their ideas about how to evaluate each expression as a means to provide opportunities for other students to hear their peers describe different strategies for solving addition facts. Students who used a *direct modeling* strategy would be called first, followed by students who used *counting on* strategies, and then by students who used *fact recall–derived facts* strategies. This particular order of strategies is intended to move the class discussion from less to more sophisticated strategies with the intent to encourage the participation and engagement from the start of students who may be using less sophisticated strategies such as *direct modeling*. The teacher intended to encourage students to draw connections between the various strategies in hopes of fostering the increased use of related doubles facts.

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Lesson Plan

In planning for this lesson, the teachers used the following learning goal:

Students will derive addition facts by using doubles as benchmarks to evaluate addition expressions.

A narrative account follows of the lesson that took place.

The teacher said, "Today, I would like you to use your math brain for some number facts problems. I will be writing them down in a way that I hope will help all of our friends. I am going to put up a number problem. I don't want you to shout out an answer. If you do have an answer, put your hand with the thumb up over your chest to show me you have an answer."

The teacher wrote 3 + 3 = on the board and read the problem aloud. The teacher turned to face the class, and when the teacher saw that most students had their thumb at their chests, she asked Sabra Ann to speak.

Sabra Ann answered, "six." The teacher asked, "Can you tell us how you got six?" Sabra Ann said, "I used my fingers." The teacher then asked Sabra Ann to show the class how she used her fingers, and she asked all the students to count along with Sabra Ann. Sabra Ann raised three fingers on one hand and counted them by ones, "one, two, three." She then raised three more fingers on her other hand, one at a time, and counted, "four, five, six."

The teacher asked, "How can I show how Sabra Ann counted on my chart so that I can remember it later?" One student offered, "Draw dots." The teacher drew dots on the board to show how Sabra Ann found her answer.

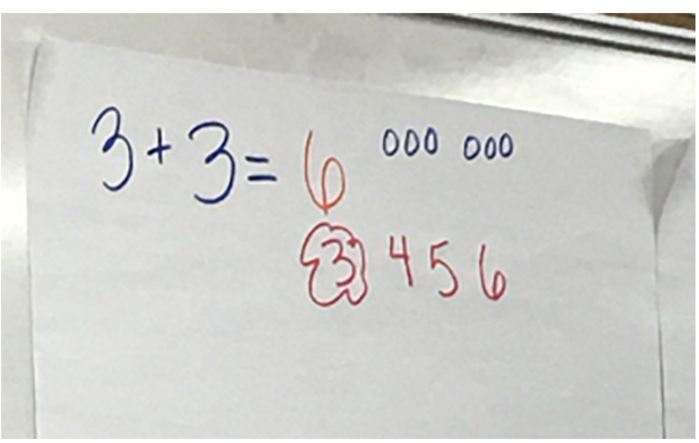

Figure 3. The teacher's notation of Sabra Ann and Jalen's strategies for 3 + 3 during the lesson.

The teacher then asked, "Did anyone find the answer a different way?" The teacher then called on Jalen to explain his strategy. Jalen said, "I put three in my head and counted up three more. The teacher notated Jalen's strategy as shown in Figure 3.

The teacher then asked the class, "How many of you just knew it was 6?" Most of the students raised their hands.

The teacher then wrote 3 + 4 = on the board and turned to face the class. The teacher waited for students to put their thumbs up. When all of the students had their thumbs up, she asked the students to turn and tell their neighbor their answer (but not their strategy).

The teacher listened to the students' answers. She waited a moment and said, "I heard a lot of sevens." The teacher then asked, "Who used a strategy like Sabra Ann?"

Nicole said, "I had four and counted up three more." The teacher asked, "Why can Nicole start with the four?" Sophia said, "Because it is better if you start with bigger number so you don't have to count up as much." Nicole said, "I actually did 4 + 3." The teacher said, "I'm going to write what you said a different way. Tell me if you agree or disagree.

The teacher then wrote 3 + 4 = 4 + 3 on a piece of chart paper. The teacher then asked the students to turn to their neighbor and tell them if they agreed or disagreed. Some of the students said, "that looks weird."

The teacher asked, "Who would like to tell me why this makes sense?" Brandon said, "It doesn't change anything, because 3 + 4 equals 7, and 4 + 3 equals 7." Quincy then said, "The equal sign means the same."

The teacher then prompted the class to read 3 +

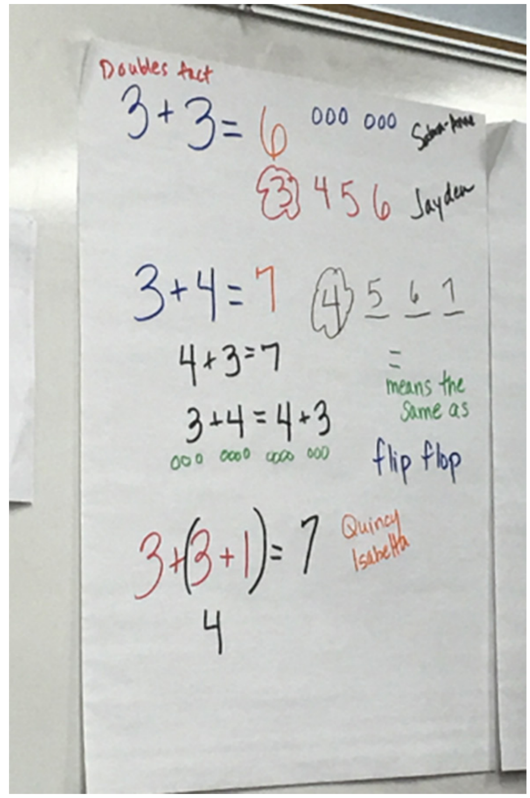

Figure 4. Teacher's notation of students' strategies for 3 + 4

4 = 7 with the phrase "the same as" in place of the equals sign. The students read, "3 + 4 means the same as 7." The teacher prompted the class to do the same thing with 3 + 4 = 4 + 3. The students then read, "3 + 4 means the same as 4 + 3." The teacher drew circles beneath each numeral to show that the left side has 7 and the right side has 7.

Quincy said, "It's like a doubles fact." The teacher responded and said, "you used a word we haven't used yet." Quincy then said, "it is doubles fact plus one."

The teacher points to 3 + 4 and said, "Quincy is trying to convince me that you can see 3 + 3 in 3 + 4. Turn to your neighbor and see how you think about that."

After students had time to talk with their neighbors, she asked, "What doubles fact is Quincy trying to convince me that I can use?" The teacher then wrote 3 + 3.

The teacher asked, "Curtis, did you agree or disagree that 3 + 3 is in 3 + 4?" Curtis said, "Disagree." The teacher then asked, "Isabelle, do you agree or disagree?" Isabelle said, "Disagree. But if you put + 1." The teacher then wrote + 1 so that it is 3 + 3 + 1. A student said, "3 + 1 makes 4." The teacher then inserted parenthesis around 3 + 1 as shown in Figure 4.

The teacher then wrote 4 + 4 on board. Students put thumbs up. The teacher then said, "Let's say the answer all together." Students all answer eight. The teacher then asked, "Do you have a special name for this?" Jade said, "Doubles fact." The teacher asked, "How could I prove it?" Jade said, "Draw circles."

The teacher then wrote 4 + 5 = on the board. Students thought about the problem and then put their thumbs up. The teacher called on several different students who said 9 was the answer.

The teacher then said, "Let's pretend I didn't know it. What could I do?" Jade said, "Count up from 5." The teacher then asked, "Who used that strategy?" Jalen said, "I did."

Joseph said "4 + 4 = 8 and then 1 more is 9." The teacher said, "Joseph is doing what Quincy did." Katie said, "Write 4 + 4 + 1 and then make = sign and write 9." The teacher said, "I'm missing some parentheses here." The teacher then placed parenthesis around 4 + 1. The teacher then said, "now could I write it this way" and wrote 4 + (4 + 1) = (4 + 4) + 1.

The teacher said, "Okay, last one." The teacher wrote 5 + 6 = on the board. The teacher called on Brandon, and he said, "11." The teacher said, "there are a lot of ways we could write it to show ways that match what we've been seeing."

Jayden said, "5 + 5 is 10 and one more is 11." The teacher wrote 5 + 5 on the board. The teacher circles 5 + 5 with her finger and asked, "What is this called?" Students answered, "Doubles."

The teacher asked, "Can we pull 5 + 5 out of 5 + 6? How can we do that?" A student said, "6 = 5 + 1." The teacher said, "So you're saying 5 + 5 + 1.

Where should I put the parentheses?" The teacher then wrote (5 + 5) + 1 = 5 + (5 + 1).

At this point, only a few minutes of math time remained. The teacher then asked, "So what have we learned today about solving addition problems? Turn to your neighbor and share something you learned today."

After students were given a minute to turn and talk, the teacher brought students back into whole group. The teacher then called on a few students to share what they had learned about solving addition facts.

Reflection

Reflections on the lesson

The teachers in the group discussed their observations from the lesson. The teachers thought it was remarkable how well students analyzed, explained, and compared their peers' strategies.

Some students, such as Jalen, continued using the *direct modeling* or counting strategies that they used in the interview. Other students used more sophisticated strategies during the lesson. For example, Quincy primarily used *direct modeling* to solve each problem during the interview, but However, during the lesson, he recognized and named doubles plus one as a strategy.

During the lesson, the teacher spent a fair amount of attention on how to notate the derived-fact strategies. She even introduced parentheses to the students. The students seemed to understand the meaning of the parentheses in this context.

Brandon recognized the concept underlying the commutative property of addition, even though he did not name the property in his explanation. Quincy offered a nice explanation for the meaning of the equals sign when the teacher displayed 3 + 4 = 4 + 3. The teacher's modeling of 3 + 4 = 4 + 3 by drawing circles to represent each numeral was an interesting way to reinforce Quincy's explanation and gave students an opportunity to see the equal sign as a relational symbol.

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What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

What's Next? is a collection of stories documenting professional development experiences shared by elementary teachers working collaboratively to study the complex process of teaching and learning mathematics. Each story in the collection describes practicing teachers studying the thinking processes of real students and using what they learn about those students to make decisions and try to help advance those students' understanding on that day.

The teachers in each story start by learning about how individual students are solving a set of mathematics problems. They use this freshly gathered knowledge of student thinking to develop near-term learning goals for students and a lesson plan tailored to specific students on that specific day. One of the teachers implements the planned lesson while the other teachers observe in real time. The teachers then gather to discuss and reflect on their observations and insights.

In these lessons, the practice of teaching is slowed way down. The stories tell of teachers who are studying student thinking and using that information to plan and implement instructional decisions at a pace that is much slower than it occurs in daily practice. The stories in this collection also depict many aspects in common with formative assessment and lesson study, both of which are a process and not an outcome.

The stories depict real situations that occurred in real time and include both successes and shortcomings. We hope that the stories may be studied and discussed by interested educators so that the lessons and ideas experiences of these teachers and instructional coaches may contribute to additional learning and sharing among other interested teachers.

Learn more about these and other stories at **http://www.teachingisproblemsolving.org/**